REGULAR AND CHAOTIC DYNAMICS OF A MATTER-WAVE SOLITON NEAR THE ATOMIC MIRROR

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The dynamics of the soliton in a self-attractive Bose-Einstein condensate under the gravity are investigated. First, we apply the inverse scattering method, which gives rise to equation of motion for the center-of-mass coordinate of the soliton. We analyze the amplitude-frequency characteristic for nonlinear resonance. Applying the Krylov-Bogoliubov method for the small parameters the dynamics of soliton on the phase plane are considered. Hamiltonian chaos under the action of the gravity on the Poincaré map are studied.

Keywords: Bose-Einstein condensate; matter-wave soliton; Krylov-Bogoliubov method; Poincaré map; Hamiltonian chaos; mean first-passage time

1. Introduction

Soliton is a localized nonlinear wave that propagate without losing its shape due to equilibrium between dispersion and nonlinearity effects.[1] Solitons appear in such physical systems as nonlinear optics, hydrodynamics and plasma waves.

The Bose-Einstein condensate (BEC) represents a giant matter-wave packet. One of the most important aspects of matter-wave packets is that they are strongly affected by gravity. In particular, they fall towards earth like a beam of ordinary atoms. Since the matter-wave packet is a superposition of macroscopic de Broglie waves of ultracold massive atoms, they are accelerated under gravity. This property of matter-waves was employed in the design of an output coupler for the first atom laser,[2] and demonstration of coherence of a freely expanding and overlapping BEC.[3]

In physics, an atomic mirror is a media which reflects neutral atoms in the similar way as the conventional mirror reflects visible light. Atomic mirrors can be made of electric fields or magnetic fields[4] electromagnetic waves[5] or just silicon wafer.[6] In the last case, atoms are reflected by the attracting tails, of the van der Waals attraction (quantum reflection).[7] Such reflection is efficient when the normal component of the wavenumber of the atoms is small or comparable to the effective depth of the attraction potential. Roughly, the distance at which the potential becomes comparable to the kinetic energy of the atom. To reduce the normal component most atomic mirrors are blazed at the grazing incidence. At grazing incidence, the efficiency of the quantum reflection can be enhanced by a surface covered with ridges.[8,9]

Recently quantum reflection of matter-waves from a solid surface has been the subject of considerable interest both from the viewpoints of basic physics and BEC applications. Specifically, matter-wave dynamics near the solid surface can be a very sensitive probe for the Casimir force.[10] Meantime, atom chips, where a BEC is stored and manipulated near the solid substrate, open up new perspectives for application.[11] Coherent acceleration of matter-wave packet falling under gravity and bouncing off a modulated magnetic mirror showed the possibility to realize the Fermi acceleration with matter-waves.[12]

This work is aimed at investigation of the dynamics of a matter-wave soliton near the solid surface under the action of a linear potential, originating from the attractive force of gravity. The effect of a solid surface is modelled by a reflecting delta-potential barrier. In real experiments such a barrier can be created by means of a laser light far-off blue-detuned from atomic transitions. The underlying mathematical model is based on the one dimensional Gross-Pitaevskii equation (GPE) for the BEC with a negative atomic scattering length, when the GPE supports self-localized solution, the so called matter-wave soliton.

The paper is organized as follows. In Sec. 2, we describe interactions between the matter-wave soliton and delta-potential barrier. In Sec. 3, the Krylov-Bogoliubov method applied to the equation of motion for the center-of-mass coordinate of the soliton. In Sec. 4, we consider the Poincaré map for nonlinear resonance. In concluding Sec. 5, we summarize our results.

2. The Model and Governing Equation

The dynamics of a BEC in the mean-field approximation at zero temperature is governed by the 3D GPE[13,14]

$$i\hbar\frac{\partial\Psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) + \frac{4\pi\hbar^2\alpha_s N}{m}|\Psi|^2\right]\Psi, \tag{1}$$

where $\Psi(\mathbf{r},t)$ is the macroscopic wave function of the condensate normalized so that $\int_{-\infty}^{\infty}|\Psi(\mathbf{r},t)|^2 d\mathbf{r} = 1$, N is the total number of atoms, m is the atomic mass, α_s is the $s-$wave scattering length (below we shall be concerned with an attractive BEC for which $\alpha_s < 0$), and

$$V(\mathbf{r}) = \frac{m}{2}\left[\omega_x^2 x^2 + \omega_\perp^2(y^2+z^2)\right] \tag{2}$$

is the axially symmetric trapping potential which provides for tight-fitting confinement in the transverse plane (y, z), as compared to free axial trapping, assuming $\omega_x^2/\omega_\perp^2 \ll 1$.

When the transverse confinement is strong enough, so that the transverse oscillation quantum $\hbar\omega_\perp$, is much greater than the characteristic mean-field interaction energy $N|\alpha_s||\Psi|^2$, the dynamics is effectively one dimensional. In this case, the 3D wave function may be effectively factorized as $\Psi(x,y,z,t) = \psi(x,t)\varphi(y,z)$, where $\varphi(y,z) = \exp\big[-(y^2+z^2)/2l_\perp^2\big]/\sqrt{\pi}l_\perp$ is the normalized ground state of the 2D harmonic oscillator in the transverse direction, with $l_\perp = \sqrt{\hbar/m\omega_\perp}$ being the corresponding transverse harmonic oscillator length. Substituting the factorized expression into the 3D GPE (1), and integrating it over the transverse plane (y,z), one derives the effective 1D GPE for an attractive BEC

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) - q_{1D}|\psi|^2\right]\psi, \tag{3}$$

where we have neglected the zero-point energy of the transverse motion $\hbar\omega_\perp$, and defined the coefficient of the 1D nonlinearity, $q_{1D} = 4\pi|\alpha_s|\hbar\omega_\perp\int_{-\infty}^{\infty}|\varphi(y,z)|^4dydz = 2|\alpha_s|\hbar\omega_\perp$ and $V(x) = m\omega_x^2x^2/2$ is the axial parabolic trap in the x direction.

Let us consider the case when the BEC falls under gravity force and bouncing off from the modulated atomic mirror. Next, we shall assume that the axially parabolic trap in the Eq. (3) can be changed by the linear potential and delta-potential barrier. As a result, the 1D GPE, taking into account the gravity can be written in the following form:

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t) - q_{1D}|\psi|^2\right]\psi. \tag{4}$$

The potential $V(x,t)$ for the 1D GPE (4) with the falling BEC under the gravity has the following form:

$$V(x,t) = V_1(x) + V_2(x,t), \tag{5}$$

$$V_1(x) = kx, \tag{6}$$

$$V_2(x,t) = V_0\delta\left[x - f(t)\right], \tag{7}$$

where $V_1(x)$ is the lineal potential, $k = mg$, g is the acceleration of gravity, $V_2(x,t)$ is the delta-potential barrier whose position is oscillating with the amplitude of external force f_0 and time dependence function given by $f(t) = f_0\sin(\gamma t + \phi)$, γ and ϕ are the frequency and the phase of the amplitude of external force.

For the purposes of further simplification, let us rewrite Eq. (4) using dimensionless variables: $t \to t\omega_\perp/2, x \to x/l_\perp, l_\perp = \sqrt{\hbar/m\omega_\perp}, l \to (l_\perp/l_g), l_g^{-3} = 2m^2g/\hbar^2, V_0 \to 2V_0/(\hbar\omega_\perp l_\perp)$, and the rescaled wave function $u \to \sqrt{2|\alpha_s|}\psi$,

$$iu_t + u_{xx} + V(x,t)u + 2|u|^2u = 0. \tag{8}$$

It is well known, in the absence of the potential term $V(x,t) = 0$, Eq. (8) gives rise to a commonly known family of soliton solutions,[15]

$$u(x,t) = 2i\eta\frac{\exp\left[-2i\xi x - 4i(\xi^2-\eta^2)t - i\phi_0\right]}{\cosh[2\eta(x-\zeta)]}, \tag{9}$$

$$\zeta = -4\xi t + \zeta_0, \tag{10}$$

where η, ξ, ζ are, respectively, the amplitude, velocity, center-of-mass coordinate and ζ_0, ϕ_0 are the initial coordinate and phase.

If we can consider the effects of the linear potential $V_1(x)$ and delta-potential barrier $V_2(x,t)$ as perturbations for the soliton

$$iu_t + u_{xx} + 2|u|^2 u = \epsilon R, \tag{11}$$

$$\epsilon R = [V_1(x) + V_2(x,t)]\, u. \tag{12}$$

Applying the conservation law of the field momentum $dP/dt = 0$ from the soliton theory[15] and taking into account the $dP/dt = i\int_{-\infty}^{\infty}(u_t u_x^* - u_x u_t^*)dx$, we finally get the following equation for the soliton center-of-mass coordinate, which has the following form:

$$\frac{d^2\zeta}{dt^2} = -2k - 8V_0\eta^2 \frac{\sinh\left[2\eta(\zeta - f(t))\right]}{\cosh^3\left[2\eta(\zeta - f(t))\right]}. \tag{13}$$

Introducing the new variable $y = \zeta - f(t)$, one obtains from Eq. (13) the following equation

$$\frac{d^2y}{dt^2} = -\frac{\partial U}{\partial y} - \ddot{f}(t). \tag{14}$$

This is the governing equation for a unit mass quasi-particle moving in the field of anharmonic potential (Fig. 1)

$$U(y) = 2ky - \frac{2V_0\eta}{\cosh^2(2\eta y)}, \tag{15}$$

and external force $F(t) = -\ddot{f}(t) = f_0\gamma^2\sin(\gamma t + \phi)$.

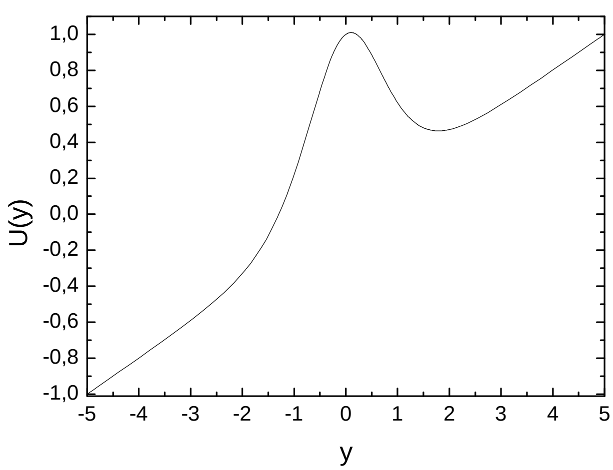

Fig. 1. The shape of the anharmonic potential given by Eq. (15) for the next parameter values: $k = -0.1$, $V_0 = -1$, $\eta = 0.5$, $y_0 = 1.78$.

Let us expand the potential $U(y)$ by a series, according to the $x = y - y_0$ degree of deviation from the point of equilibrium, with fourth order of approximation inclusively, then, the equation of motion (14) can be written as follows:

$$\ddot{x} + \omega^2 x + \alpha x^2 + \beta x^3 = f_0 \gamma^2 \sin(\gamma t + \phi), \tag{16}$$

with the coefficients

$$\omega^2 = 16 V_0 \eta^3 \mathrm{sech}^2\left(2\eta y_0\right)\left[3\mathrm{sech}^2\left(2\eta y_0\right) - 2\right], \tag{17}$$

$$\alpha = 64 V_0 \eta^4 \tanh\left(2\eta y_0\right) \mathrm{sech}^2\left(2\eta y_0\right)\left[1 - 3\mathrm{sech}^2\left(2\eta y_0\right)\right], \tag{18}$$

$$\beta = (32/3) V_0 \eta^5 \mathrm{sech}^6\left(2\eta y_0\right)\left[26\cosh(4\eta y_0) - \cosh(8\eta y_0) - 33\right], \tag{19}$$

where ω is the frequency of the quasi-particle, α and β are the anharminic coefficients, γ is the frequency of the external force and ϕ is the phase.

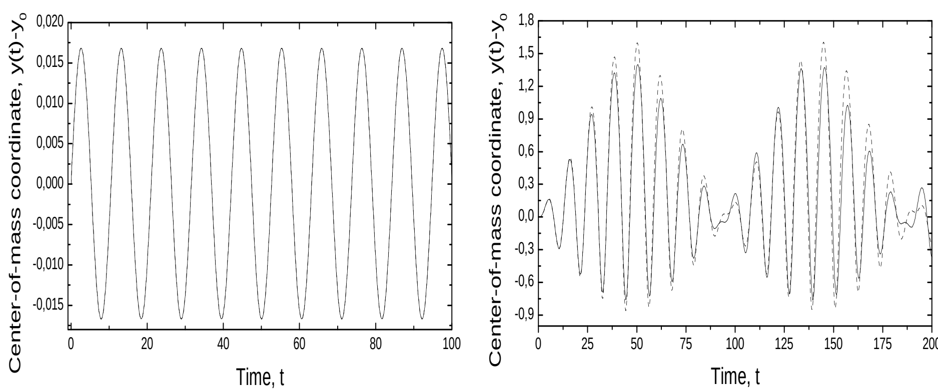

Fig. 2. The dynamics of the center-of-mass coordinate of soliton. Numerical simulation of GPE (8) (solid line) and governing equation (14) (dashed line). Left panel: The case when the delta-potential barrier do not oscillating. Parameters are: $V_0 = -1, \eta = 0.5, y_0 = 1.78, \omega = 0.596, \alpha = -0.273, \beta = 0.081, a = 0.39, f_0 = 0$. Right panel: Delta-potential barrier as perturbation of the external force. Parameter values: $\gamma = \omega$, $f_0 = 0.1$ (beats).

It is interesting to consider the dynamics of the interaction between the soliton and the oscillating surface in the proximity of the resonance, which is illustrated in the Fig. 2. The left panel of Fig. 2, illustrates the cases when the interacting surface does not oscillating, the soliton oscillates harmonically with the period $T = 2\pi/\omega\left[1 + (5a^2/12\omega^4 - 3\beta/8\omega^2)a^2 + o(a^2)\right]$, as described in Ref. 18. Thus, near a position of stable equilibrium, a system executes harmonic oscillations. As can be seen in the right panel of Fig. 2, the dependence of the amplitude a of the forced oscillations on the frequency of the external force has the characteristic resonance shape: The nearer the frequency of the external force to the natural frequency ω, the more the external force rocks the system. The phase ϕ of the forced oscillations undergoes a jump of $-\pi$ as γ passes thought the resonance frequency ω. When γ is near ω, beats are observed, i. e., the amplitude of the quasi-particle alternately waxes, when the relation of the phases of the quasi-particle and the external force is such that the external force rocks the quasi-particle, communicating energy to it and wanes, when the relation between the phases changes in such a way that the external force brakes the quasi-particle. The closer frequencies γ and ω, the more slowly the phase relation changes and the larger the period of the beats. As $\gamma \to \omega$, the period of the beats approaches infinity.

In order to estimate the actual values of the time and space units, we shall provide the experimental parameters from Ref. 16. The current experiment considers a single soliton of the lithium condensate. The $s-$wave scattering length, at the value of the magnetic field $B = 425$ G (which was used to make the atomic interaction attractive, via Feshbach resonance), was $a_s = -0.21$ nm. With the mass of a ^{7}Li atom, $m = 11.65 \times 10^{-27}$ kg, we have the following time and space units: $\omega_x^{-1} \simeq 3 \times 10^{-3}$ s, $l_\perp = \sqrt{\hbar/m\omega_\perp} \simeq 12\ \mu m$, the trap's aspect ratio being $\omega_x/\omega_\perp \simeq 7 \times 10^{-2}$.

3. The Dynamics in the Potential Field

The case of a small nonlinearity turns out to be more complicated, strange though it may seem, if, perhaps, a more interesting one, as is demonstrated by a curious example in Ref. 17. When the anharmonic terms in forced oscillations of a system are taken into account, the phenomena of resonance acquire new properties. Let $\gamma = \omega + \Delta$, with small Δ, i. e. γ be the resonance value. Strictly speaking, when nonlinear terms are included in the equation of the free oscillations, the term higher order in the amplitude of external force (such as occur if it depends on the displacement x) should also be included. We shall omit these terms merely to simplify the formulae, i. e. they do not affect the qualitative results. As well known,[18] in the linear approximation, the amplitude a is given near resonance, as a function of the amplitude f_0 and frequency γ of the external force, which we write as $a^2\varepsilon^2 = f_0^2/4\omega^2$. The nonlinearity of the oscillations results in the appearance of an amplitude dependence of the eigenfrequency, which we write as $\omega + (3\beta/8\omega)a^2$. Accordingly, we replace ω by $\omega + (3\beta/8\omega)a^2$ (or, more precisely, in the small difference $\gamma - \omega$). With $\Delta = \gamma - \omega$, the resulting equations is

$$a^2\left(\Delta - \frac{3\beta}{8\omega}a^2\right)^2 = \frac{f_0^2}{4\omega^2}. \tag{20}$$

Eq. (20) is a cubic equation in a^2, and its real roots give the amplitude of the external forced oscillations. Let us consider how this amplitude depends on the frequency of the external force for a given amplitude f_0 of that force. As f_0 increases, the curve changes its shape, though at first it retains its single maximum, which moves to positive Δ if $3\beta/8\omega > 0$. At this stage only one of the three roots of Eq. (20) is real. When f_0 reaches a certain value $f_{\rm th}$ (to be determined below), however, the nature of the curve changes. For all $f_0 > f_{\rm th}$ there is a range of frequencies in which Eq. (20) has three real roots. In the absence of friction, the damping coefficient is zero. Consequently the Fig. 3 indicates that in our case the damping coefficient, which affects the knee of the curve, is zero leading to the branches of the amplitude-frequency characteristic receding into infinity.

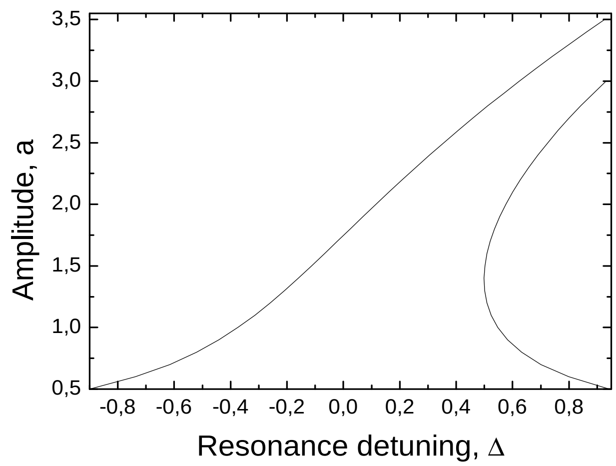

Fig. 3. Amplitude-frequency characteristic for nonlinear resonance at $\omega = 0.596, \beta = 0.081, f_0 = 0.55$.

It is widely known that nonlinear oscillating systems with weakly nonlinearity can be studied by the perturbation theory methods. Let us consider the dynamics of the soliton in one dimension, and writing the Eq. (16) in the form:

$$\ddot{x} + x = \varepsilon Q(x, \nu\tau), \tag{21}$$

where $\varepsilon Q(x, \nu\tau) = -\delta x^2 - \lambda x^3 + \chi \sin(\nu\tau)$ is the periodic function with respect to $\nu\tau$ with period 2π, $\dot{x} = dx/d\tau, \tau = \omega t, \delta = \alpha/\omega^2, \lambda = \beta/\omega^2, \chi = f_0\nu^2, \nu = \gamma/\omega$ are the dimensionless variables, ε is the small positive parameter, indicating the smallness of the function $\varepsilon Q(x, \nu\tau)$ with regard to the linear term (the order of smallness of the terms in this equation is determined, so that with $\varepsilon \to 0$ there is the case for linear harmonic oscillations). In this case, we are considering the main resonance, i. e. $\nu = 1$. Higher order resonance appear when $\omega \approx (n/r)\gamma$, where n and r are integers. Taking into account the smallness of nonlinear terms ($x \ll 1$) with regard to the linear terms for solving Eq. (21) in zero-order approximation $x(\tau)$ can be chosen as[19]:

$$x(\tau) = b(\tau) \cos\left[\sigma(\tau)\right], \tag{22}$$

where $\sigma(\tau) = \nu\tau + \theta(\tau)$ and b, θ are slowly varying functions of τ. Using the Krylov-Bogoliubov method (KBM) for the small parameters[19] it is possible to obtain the following coupled equations for b and θ:

$$\frac{db}{d\tau} = -\frac{\chi}{2\nu} \cos\theta, \tag{23}$$

$$\frac{d\theta}{d\tau} = \frac{\rho}{2\nu} + \frac{3\mu b^2}{8\nu} + \frac{\chi}{2\nu b} \sin\theta, \tag{24}$$

where $\rho = 1 - \nu^2, \rho << 1, \mu = \lambda\left[1 - 10\delta^2/(9\lambda)\right]$.

The equations (23)-(24) can be transformed into Hamiltonian form by changing the new variable $b = \sqrt{p}$. The Hamilton's equations are $dp/dt = -\partial H/\partial\theta, d\theta/dt = \partial H/\partial p$, with Hamiltonian $H = (\chi\sqrt{p}/\nu)\sin\theta + \rho p/(2\nu) + 3\mu p^2/(16\nu)$. Hamiltonian is conserved quantity, i. e. H is the integral of motion for system (23)-(24). Phase trajectories corresponding to different values of H at fixed χ and ρ are shown in Fig. 4. From qualitative analysis of the system (23)-(24) it can be revealed that there exists a separatrix $H = 0$, which separates finite trajectories corresponding to nonlinear resonances from infinite ones. Figure 4 illustrates the quasi-particle motion on the phase plane at threshold Hamiltonian $H_{\rm th} = 0.33$ corresponds to the separatrix. When the value of Hamiltonian is above threshold, i. e. $H = 0.4$, then we observe the quasi-particle can leave the potential well, performing nonlinear oscillations.

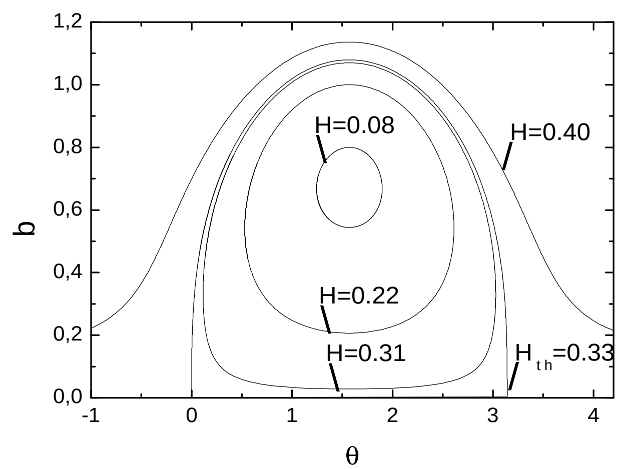

Fig. 4. The phase trajectories for system (23)-(24). The curve for $H_{\rm th} = 0.33$ corresponds to the separatrix; the curve for $H = 0.4$ corresponds to the drift trajectory. Parameters are: $\chi = 0.1, \nu = 1, \rho = 0, \mu = -0.423$.

4. The Dynamics on the Poincaré Map

In this section, we will describe the criterion of chaotic oscillations in the problem of quasi-particle motion in the potential field $U(y)$ under the action of periodic force $F(t)$ (see section 1). The force, which leads to such kind of the motion, is potential. It is known,[20,21] that after a transient process there is a steady state in the system (16), i. e. $x(t) \sim a\cos(\gamma t + \vartheta_0)$, where a and ϑ_0 are amplitude and initial phase of oscillations respectively. One can obtain by Lindstedt method[18,22] the relation between $\alpha, \beta, \gamma, \omega$ and amplitude a: $F(\alpha, \beta, \gamma, \omega, a) = f_0$, where F is some function. This relation may be considered as an equation for threshold amplitude

$f_{0\text{th}}$. As such, substituting this solution to Eq. (16) and equating the coefficients of trigonometric functions we finally obtain the following condition for breaking of the bound state:

$$f_0 \geq f_{0\text{th}} = a_{\text{th}} \left\{ \left[(\omega/\gamma)^2 - 1 + 3\beta a_{\text{th}}^2/4\gamma^2 \right]^2 + (3/4)\left(\alpha a_{\text{th}}/\omega\gamma^2\right)^2 \right\}. \tag{25}$$

This criterion is determine the transition boundary from periodic motion to the chaotic one. It is important to draw a line between the systems of the damped oscillations and the ones without such. In the systems without the damped oscillations or weakly damped oscillations the Poincaré map of the chaotic motion often has a form of disordered clusters of points on the Poincaré map. Such motions are called stochastic. Under the action of external force near trajectories with the pe-

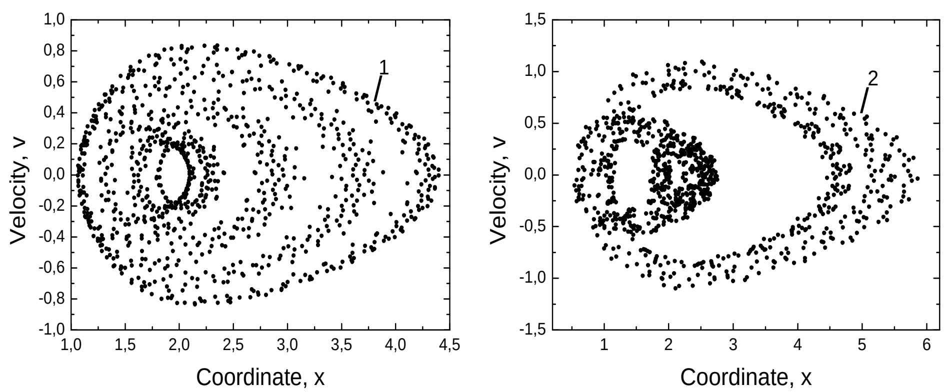

Fig. 5. The Poincaré map of Eq. (16). Left panel: The set of points 1 corresponds to the quasi-periodic motion with the next parameters: $\omega = 0.596, \alpha = -0.273, \beta = 0.081, f_0 = 0.45$. Right panel: The set of points 2 corresponds to the chaotic motion and going away to $+\infty$ with $f_{0\text{th}} = 0.55$.

riod $T = nT_{\text{ext}}$, where $T_{\text{ext}} = 2\pi/\omega$ and n is integer, nonlinear resonance occurs.[23] Figure 5 illustrates the Poincaré map, as can be seen in the left panel, the set of points 1 represents quasi-periodical motion, the separatrix and stationary points of the Poincaré map appear due to nonlinear resonance. The width of the separatrix of the nonlinear resonance increases with growing of f_0, and at some values of f_0 the overlap of neighboring resonances occurs. In Fig. 5, we show that if the amplitude of the external force changes from $f_0 = 0.45$ to $f_0 = 0.5408$, then the quasi-periodic motion of the quasi-particle occurs. As can be seen in the right panel of Fig. 5, the chaotic motion of the quasi-particle occurs as the value of the amplitude of the external force approaches the threshold value of $f_{0\text{th}} = 0.55$. This threshold amplitude separates the quasi-periodic motion from the chaotic motion of the quasi-particle. The set of points 2 represents chaotic motion of the quasi-particle. As a result, the motion of the quasi-particle with the threshold amplitude on the Poincaré map becomes random, i. e. Hamiltonian chaos appears in the system.[23,24]

The typical picture of escape of the quasi-particle from the potential well under the action of resonance force is shown in the Fig. 6. On the other hand, the quasi-particle leaves the potential well when its kinetic energy becomes comparable with the value corresponding to the difference between the potential well bottom and the separatrix. When the amplitude of the external force reaches its threshold, the quasi-particle accelerates to the boundary to leave the potential well. Figure 6 illustrates the mean-first passage time[25] (MFPT) of the quasi-particle, as a result of numerical integration of governing equation (14) at $f_{0\mathrm{th}} = 0.55$.

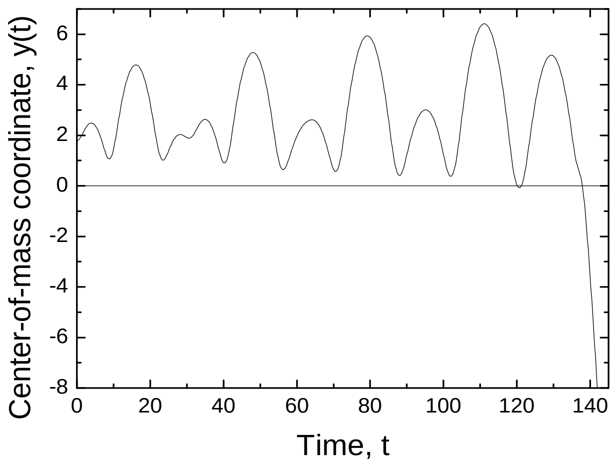

Fig. 6. The result of numerical integration of Eq. (14) for the next parameters: $V_0 = -1, \eta = 0.5, y_0 = 1.788, \omega = 0.596, \alpha = -0.273, \beta = 0.081, f_{0\mathrm{th}} = 0.55$.

5. Conclusion

We have studied the nonlinear effect the matter-wave soliton interacting with the delta-potential barrier by means of nonlinear mechanics and numerical simulations. The Krylov-Bogoliubov method provides a framework to understand the dynamics of the quasi-particle on the phase plane. Effects of nonlinear resonances, are studied by perturbation theory. By applying the Poincaré map, we showed as the amplitude of the external force approaches to the threshold value, a Hamiltonian chaos can be observed in the system. The developed model predicts that the quasi-particle, being on the bottom of the potential well, begins to scatter through resonances and escapes the potential well, increasing stochastically its energy. The results can be useful in development of new methods aimed at the delta-potential barrier by scattering solitons on them.

Two related physical phenomena have recently been observed: quantum states of ultra-cold neutrons in the gravitational field above a flat mirror, and quantum states of cold neutrons in an effective centrifugal potential in the vicinity of a concave mirror.[26] Obtained results can be useful for experiments with ultra-cold neutrons and ultra-cold quantum gases.

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